MA 141-004

Thursday, March 28

- today:
  - 4.2 (definite integrals, areas)
  - 4.3 (Fundamental Theorem of Calculus)

4.2 EXERCISES:

(5) $$\int_1^3 (2-3x+x^2)\,dx = \left[2x - 3\frac{x^2}{2} + \frac{x^3}{3}\right]_1^3$$

$$= \left(2(3) - \frac{3}{2}\cdot(3^2) + \frac{(3)^3}{3}\right) - \left(2(1) - \frac{3}{2}(1)^2 + \frac{(1)^3}{3}\right)$$

$$= 6 \boxed{-\frac{27}{2}} + 9 - 2 + \boxed{\frac{3}{2}} - \frac{1}{3}$$

$$= 13 - 12 - \tfrac{1}{3} = \underbrace{\tfrac{2}{3}}$$

(6) $$\int_0^{\pi/2} (2\sin x + 3)\,dx$$

$$= \Big[2\cdot(-\cos x) + 3x\Big]_0^{\pi/2}$$

$$= \left(-2\cos \tfrac{\pi}{2} + 3\left(\tfrac{\pi}{2}\right)\right) - \left(-2\cos 0 + 3(0)\right)$$

$$= -2\cancel{(0)} + \tfrac{3\pi}{2} + 2(1) - \cancel{0}$$

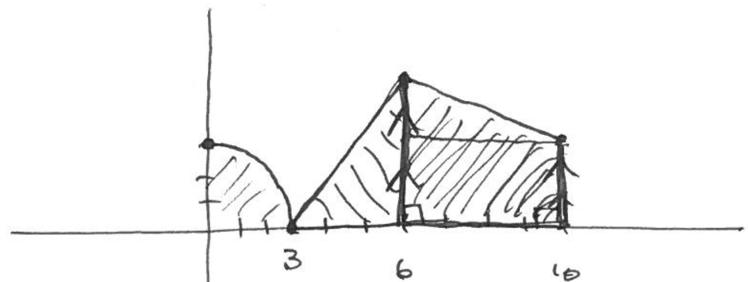
$$= 2 + \tfrac{3\pi}{2} \approx \underline{\qquad}$$

11 $\displaystyle\int_{\pi/2}^{\pi/2} \sqrt[3]{\sec^4(8\theta) - \tan^7(5\theta)}\, d\theta = 0$

9 $f(x) = \begin{cases} \sqrt{9-x^2} & 0 \le x \le 3 \\ \frac{5}{3}x - 5 & 3 < x \le 6 \\ -\frac{1}{2}x + 8 & 6 < x \le 10 \end{cases}$

$y = \sqrt{9-x^2}$
$y^2 = 9 - x^2$
$x^2 + y^2 = 9$

$\displaystyle\int_0^{10} f(x)\, dx$



$\displaystyle = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx + \int_6^{10} f(x)\,dx$

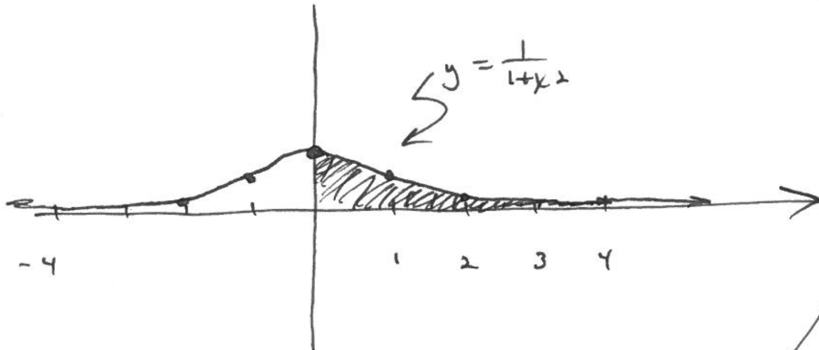
$\displaystyle = \int_0^3 \sqrt{9-x^2}\,dx + \int_3^6 \left(\tfrac{5}{3}x - 5\right)dx + \int_6^{10}\left(-\tfrac{1}{2}x + 8\right)dx$

$= \frac{1}{4}\left(\pi \cdot 3^2\right) + \frac{1}{2}(3)(5) + \frac{1}{2}(5+3)(4)$

$= \frac{9\pi}{4} + \frac{15}{2} + 16$

$$\int_0^4 \frac{8}{1+x^2}\,dx = 8\int_0^4 \frac{1}{1+x^2}\,dx$$

$y = \frac{1}{1+x^2}$



$$= 8\tan^{-1}(x)\Big]_0^4$$

$\tan\theta = 4$

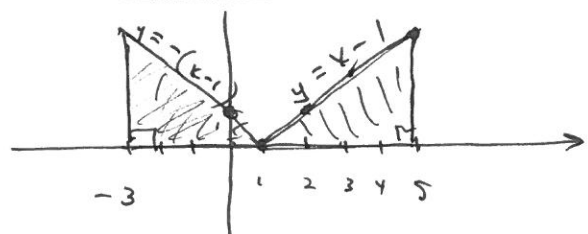
$\tan(1.3258) = 4$

$$= 8\left[\tan^{-1}(4) - \tan^{-1}(0)\right]$$

$$\approx 8\left[1.3258 - 0\right] \approx 8(1.3258)$$

$$\int_{-3}^{5} |x-1|\,dx$$

$y = |x-1|$



$$= \int_{-3}^{1} (-x+1)\,dx + \int_1^5 (x-1)\,dx$$

4.3 Fundamental Theorem of Calculus

Earlier in this course, we found it difficult to compute derivatives using only the formal definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Later we found several methods of finding derivatives that made the process more tractable. We now find ourselves at a similar place in the computation of the definite integral. To apply the definition we need to set up Riemann sums, simplify using various properties alongside summation equations proved by mathematical induction $\left(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}; \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}; \ldots \right)$, and take a limit to find the area. Unfortunately, similar summation formulas are not available for many of the functions we encounter. As a result of this challenge, we introduce and validate more efficient methods of finding definite integrals and in the process make a fundamental connection between differential and integral calculus that is unrivaled in scientific importance.

The Fundamental Theorem of Calculus does just that – thanks to the ground breaking work of Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716). It establishes the inverse relationship between differentiation and integration – allowing the study of calculus to develop into a systematic mathematical method.

Theorem 4. The Fundamental Theorem of Calculus

Let f be continuous on $[a,b]$.

$G(t)\Big]_a^x = G(x) - G(a)$

$\#$

Part 1: The function G defined by

$$G(x) = \int_a^x f(t)\,dt \qquad \textit{for all } x \textit{ in } [a,b]$$

is an antiderivative of f on $[a,b]$.

Part 2: If F is any antiderivative of f on $[a,b]$, then

$F(x)\Big]_a^b =$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$G(x) = \int_8^x (3t^2 + t)\,dt$$

find $\dfrac{d(G(x))}{dx}$:

$$\frac{d(G(x))}{dx} = \frac{d\left[\int_8^x (3t^2+t)\,dt\right]}{dx}$$

$$\frac{d(G(x))}{dx} = \frac{d\left(\left[t^3 + \frac{t^2}{2}\right]_8^x\right)}{dx}$$

$$= \frac{d\left[\left(x^3 + \frac{x^2}{2}\right) - \left(8^3 + \frac{8^2}{2}\right)\right]}{dx}$$

$$= 3x^2 + x$$

$$\int_e^{e^2} \frac{7}{3x}\,dx = \frac{7}{3}\int_e^{e^2} \frac{1}{x}\,dx$$

$$= \frac{7}{3}\ln|x| \;\ldots\ldots$$

F.T.O.C (part 1)

find $\underline{G'(x)}$ $\underline{\text{or}}$ $\frac{d(G(x))}{dx}$

$$\frac{d(G(x))}{dx} = \frac{d\left(\int_2^x \frac{1}{t^3+1}\,dt\right)}{dx}$$

$$= \frac{1}{x^3+1}$$

$$F(x) = \int_x^1 \cos(\sqrt{t})\,dt$$

find $\dfrac{d(F(x))}{dx}$:

$$\frac{d(F(x))}{dx} = \frac{d\left(\int_x^1 \cos\sqrt{t}\,dt\right)}{dx}$$

$$= \frac{d\left(-1\int_1^x \cos\sqrt{t}\,dt\right)}{dx}$$

$$= -1 \cdot \cos\sqrt{x}$$

$$G(x) = \int_1^{x^3} 5 \cdot \ln t \cdot dt$$

find $\dfrac{d(G(x))}{dx}$:

$$\boxed{u = x^3}$$

$$\frac{d(G(x))}{dx} = \frac{d\left(\int_1^{x^3} 5 \cdot \ln t \cdot dt\right)}{dx}$$

$$= \frac{d\left(\int_1^{u} 5 \cdot \ln t \cdot dt\right)}{du} \cdot \frac{du}{dx}$$

$$= 5 \cdot \ln u \cdot 3x^2$$

$$= \underbrace{5 \cdot \ln x^3} \cdot (3x^2)$$

$$= 15 \cdot x^2 \cdot \ln(x^3)$$

4.3.1 Exercises

1. Find $\frac{dF}{dx}$ given $F(x) = \int_1^x (t^2+t)\,dt$ $\longrightarrow$ x^2+x

2. Find $\frac{dF}{dx}$ given $F(x) = \int_x^\pi 3\cos t\,dt$ $\longrightarrow$ $-3\cos x$

3. Find $\frac{dF}{dx}$ given $F(x) = \int_0^{x^2} (2s-1)\,ds$ $\longrightarrow$ $(2x^2-1)\cdot 2x$

4. Find $\frac{dF}{dx}$ given $F(x) = \int_{\frac{\pi}{6}}^{x} (\tan 2\theta + 3\theta)\,d\theta$

In **Exercises 5 - 7** find $G'(x)$.

5. $G(x) = \int_x^{\ln 3} (2e^t - 8t + 1)\,dt$

6. $G(x) = \int_2^{x^4} 5\sqrt{t}\,dt$

7. $G(x) = \int_{\pi/4}^{x} (1+\cos 2\theta)\,d\theta$

In **Exercises 8 - 18** evaluate each definite integral. You may use the Fundamental Theorem of Calculus or a geometric understanding of the integral.

8. $\int_{-2}^{4} (|3x-2|+1)\,dx$

9. $\int_0^3 \sqrt{9-x^2}\,dx$

10. $\int_0^4 (x^2+1)\,dx$

11. $\int_0^{\frac{\pi}{6}} \underline{\sec\theta\tan\theta}\,d\theta$ $\longrightarrow$ $\sec\theta\Big]_0^{\pi/6} = \sec\frac{\pi}{6} - \sec 0$

$\left(\frac{2}{\sqrt{3}}\right) - 1$

$\frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} - 1$

$\frac{2\sqrt{3}}{3} - 1$

12. $\int_1^8 6\sqrt[3]{x}\,dx$

13. $\int_{-1}^{1} \underline{(2t-3)^2}\,dt$

$\int_{-1}^{1} (4t^2 - 12t + 9)\,dt$

$= \left[\frac{4}{3}t^3 - 6t^2 + 9t\right]_{-1}^{1}$

14. $\int_1^{16} \frac{u+1}{\sqrt[4]{u}}\,du$

15. $\int_0^\pi (1-2\cos r)\,dr$

16. $\int_e^1 (\frac{1}{x} - 5x)\,dx$

17. $\int_0^3 (3^x + 4x)\,dx$

18. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 2\theta\,d\theta$

Last update: December 7, 2017

14.) $\displaystyle\int_1^{16} \frac{u+1}{\sqrt[4]{u}}\,du$

$$= \int_1^{16} \left(\frac{u}{\sqrt[4]{u}} + \frac{1}{\sqrt[4]{u}}\right) du$$

$$= \int_1^{16} \left(u^{3/4} + u^{-1/4}\right) du$$

$$= \left.\frac{u^{7/4}}{7/4} + \frac{u^{3/4}}{3/4}\right]_1^{16}$$

$$= \left.\frac{4}{7}u^{7/4} + \frac{4}{3}\cdot u^{3/4}\right]_1^{16}$$

$$= \left(\frac{4}{7}(16)^{7/4} + \frac{4}{3}(16)^{3/4}\right) - \left(\frac{4}{7}(1)^{7/4} + \frac{4}{3}(1)^{3/4}\right)$$

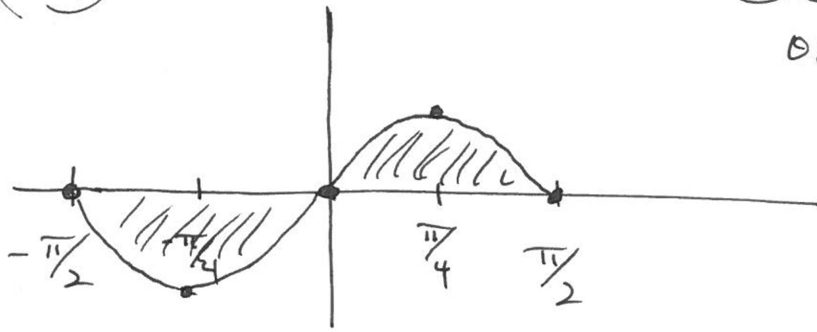
$$\left(\frac{4}{7}(2^7) + \frac{4}{3}(2^3)\right) - \left(\frac{4}{7}(1) + \frac{4}{3}(1)\right)$$

18.) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^3(2\theta)\, d\theta = 0$

(9)

$y = \sin^3(2x)$

ODD function



MA 141-004

TEST #3 RESULTS

(after +8)

A's 4

B's 5

C's 9

D's 2

F's 6

average: 72.58

1. Use Riemann Sums (right endpoints and $n = 4$ rectangles of equal width) to approximate the area of the shaded region under the curve $f(x) = 4x - x^2$. Use the limit definition of definite integral to determine the exact area. (see graph A)

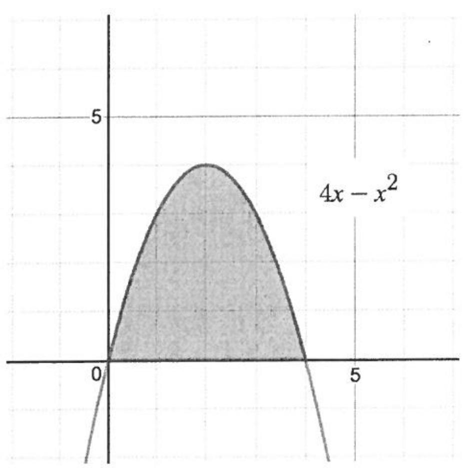
2. Let f and g be continuous functions on the interval $[0, 10]$ such that $\int_0^4 f(x)dx = -8$, $\int_4^{10} f(x)dx = 18$, and $\int_0^{10} g(x)dx = 30$. Use the Properties of Definite Integrals to determine

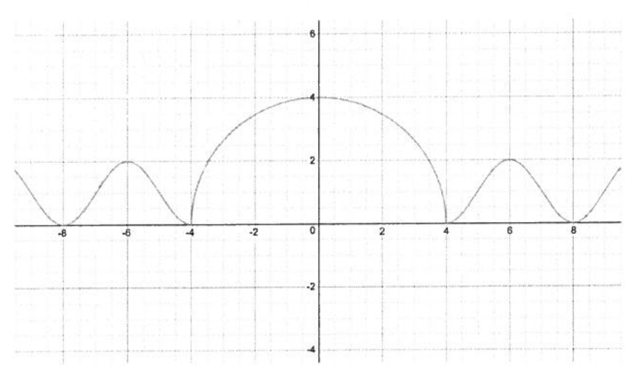
$$\int_0^{10} \left[\frac{1}{2}f(x) - g(x)\right] dx.$$

3. Use the geometric understanding of the integral and the fact $\int_4^8 (-\cos(\frac{\pi x}{2}) + 1)dx = 4$, to evaluate $\int_{-8}^8 f(x)dx$ where $f(x) = \begin{cases} \sqrt{16 - x^2}, & \text{for } |x| \leq 4 \\ -\cos(\frac{\pi x}{2}) + 1, & \text{for } 4 < |x|. \end{cases}$ (see graph B)

4. Use the Fundamental Theorem of Calculus Part 2 to determine the definite integrals

$$\int_0^{\pi} [\cos(x) + \sin(x)]\, dx \qquad \int_e^{e^2} \left[\frac{7}{3x} + 2\right] dx$$





Definition 1. **Definite Integral of a Continuous Function on $[a, b]$**

Let $f(x)$ be defined and continuous on the interval $[a, b]$. For each partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

of the interval $[a, b]$ into n equal parts, the length of each subinterval is $\Delta x = \dfrac{b-a}{n}$ and each $x_i = a + i\Delta x, i = 1, 2, \cdots, n$. The **definite integral** *of f on $[a, b]$, denoted $\int_a^b f(x)\,dx$, is defined by*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

provided this limit exists.

Theorem 2. **Formulas for Sums**

2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Properties of the Definite Integral

Let a, b and c be real numbers with $a < b$ and f and g be continuous functions. Then

1. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
2. $\displaystyle\int_a^a f(x)\,dx = 0$
3. $\displaystyle\int_a^b c\,dx = c(b-a)$
4. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
6. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Theorem 4. **The Fundamental Theorem of Calculus**

Let f be continuous on $[a, b]$.

Part 2: If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$