MA141-004

Tuesday, March 19

$$\underline{\underline{f'(c)}} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

$$0 = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} - \frac{f'(c)\cdot\Delta x}{\Delta x}$$

$$0 = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c) - f'(c)\cdot\Delta x}{\Delta x}$$

$$\Delta y = \underbrace{f'(c)\cdot\Delta x}_{dy} + \underbrace{\varepsilon\cdot\Delta x}_{\text{error}} \to 0$$

as $\Delta x \to 0$

$\Delta y \approx dy$

$(dy = f'(c)\cdot\Delta x)$

3.6 Differentials and Antiderivatives

Recall that Δy represents the total change of a function $y = f(x)$ as x varies from c to $c + \Delta x$ for some value c. We will use the definition of the derivative to split the **total change** Δy into two parts, the **principal part** which is related to $f'(x)$, and the **error part**. We begin with a theorem.

Theorem 1. *Let the function f be defined on the interval (a, b). If f is differentiable at $c \in (a, b)$ then*

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c) - f'(c)\Delta x}{\Delta x} = 0 \tag{11}$$

Proof. Although we can appeal to Lemma 1.3.1 to prove this formula let us show the limit exists by direct calculation. Thus suppose $f'(c)$ exists. Then

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c) - f'(c)\Delta x}{\Delta x} &= \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} - \lim_{\Delta x \to 0} \frac{f'(c)\Delta x}{\Delta x} \\
&= f'(c) - f'(c) \\
&= 0
\end{aligned}$$

□

We can conclude from this theorem that if a function f is differentiable at $x = c$, then the quantity that appears in Limit Formula (11) is a **function of** Δx, traditionally denoted $\varepsilon = \varepsilon(\Delta x)$:

$$\varepsilon = \frac{f(c + \Delta x) - f(c) - \cancel{f'(c)\Delta x}}{\cancel{\Delta x}} \to 0 \quad \text{as } \Delta x \to 0 \tag{12}$$

By Theorem 1 the function ε has the property $\lim_{\Delta x \to 0} \varepsilon(\Delta x) = 0$. Solving Equation (12) for $\Delta y = f(c + \Delta x) - f(c)$ we have the following.

Theorem 2. *If f is differentiable at c then $\Delta y = f(c + \Delta x) - f(c)$ satisfies*

$$\boxed{\Delta y = (f'(c) + \varepsilon)\Delta x} \tag{13}$$

$\Delta y \approx f'(c) \cdot \Delta x + \varepsilon \cdot \Delta x$

where the function ε has the property that

$$\lim_{\Delta x \to 0} \varepsilon(\Delta x) = 0$$

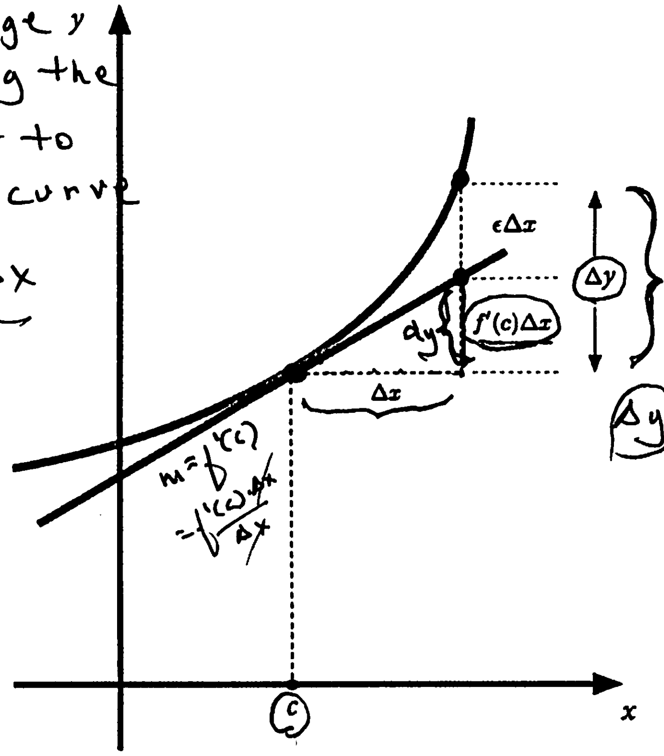
$\Delta y \approx f'(c) \cdot \Delta x$

The term $f'(c)\Delta x$ in Equation (13) is called the **principal part** of Δy, since this is the part that is determined by the derivative $f'(c)$, and the term $\varepsilon\Delta x$ is called the **error part** of Δy. The two parts of Δy are shown in the following figure.

Δy = the exact change in y on the curve itself.

dy = the change in y along the tangent to the curve

$dy = \underbrace{f'(c) \cdot \Delta x}$



if $\Delta x \to 0$ ("small") then $dy \approx \Delta y$

$\Delta y = \underbrace{f'(c) \cdot \Delta x}_{dy} + \boxed{\underbrace{\varepsilon \cdot \Delta x}_{\text{error}}}$

Figure 82: $\Delta y = f'(c)\Delta x + \varepsilon\Delta x$

REMARK: There are two notations involving ε and Δx which look very similar but have quite different meanings. This first comes from the definition of ε as a *function* of Δx. In this case we write $\varepsilon = \varepsilon(\Delta x)$. The second is the product of ε and Δx, $\varepsilon \cdot \Delta x = \varepsilon\Delta x$. This product is the *error* in using $dy = f'(c)\Delta x$ to approximate Δy.

Let's look at two examples to see what ε looks like in specific cases.

Example 34. *Let $f(x) = x^2$ with $f'(x) = 2x$. Find the function ε that appears in the Formula (13) above.*

$$f(x) = x^3 \qquad f'(x) = 3x^2$$

$$\Delta y = f(x+\Delta x) - f(x) \qquad \Delta y = (f'(c) + \varepsilon)\,\Delta x$$

$$\Delta y = (x+\Delta x)^3 - x^3$$

$$\Delta y = \left[\cancel{x^3} + 3x^2\cdot\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right] - \cancel{x^3}$$

$$\Delta y = \underbrace{3x^2\cdot\Delta x} + \underbrace{3x(\Delta x)^2} + \underbrace{(\Delta x)^3}$$

$$\to \Delta y = \left[3x^2 + \overset{\varepsilon(\Delta x)}{\boxed{3x\cdot(\Delta x) + (\Delta x)^2}}\right]\cdot\Delta x$$

$$\Delta y = \left[3x^2 + \varepsilon(\Delta x) \to 0\right]\cdot\Delta x \qquad \varepsilon = 3x(\Delta x) + (\Delta x)^2$$

as $\Delta x \to 0$

$$\Delta y \approx \left[f'(x)\cdot\Delta x\right] \approx 3x^2\cdot\Delta x$$

Example 38. *A spherical water tank of radius 5 meters is to be painted. The thickness of the paint on the sphere will be 3 mm. Use differentials to approximate the amount of paint in m^3 that it will take to paint the water tank. How many gallons of paint are required?*

Solution: The volume of the spherical tank is

$$V(r) = \frac{4}{3}\pi r^3$$

$$dV = V'(r)\,dr = (4\pi r^2)\,dr = (4\pi r^2)\,\Delta r$$

Using $r = 5$ m and $\Delta r = 3$ mm $= 0.003$ m yields

$$dV = (4\pi \cdot 5^2)\,0.003 \approx 0.9425\,\text{m}^3$$

There are 264 gallons in 1 cubic meter. Hence it will take approximately 249 gallons of paint to cover the water tank.

Example 36. *If $f(x) = x^3$ compute its differential df and approximate the change in f when x changes from 8 to 8.1. Using Example 35 find the error $\varepsilon\Delta x$ in using df to approximate Δf. Then find an approximation for the relative change of f as x changes from 8 to 8.1.*

Last update: November 2, 2018

$(dy = df)$

$f(x) = x^3$

$f'(x) = 3x^2$

$df = f'(c) \cdot \Delta x$

$df = (3x^2) \cdot \Delta x$ $\quad x = 8$, $\Delta x = .1$

Δx "small" $(\Delta x = .1)$

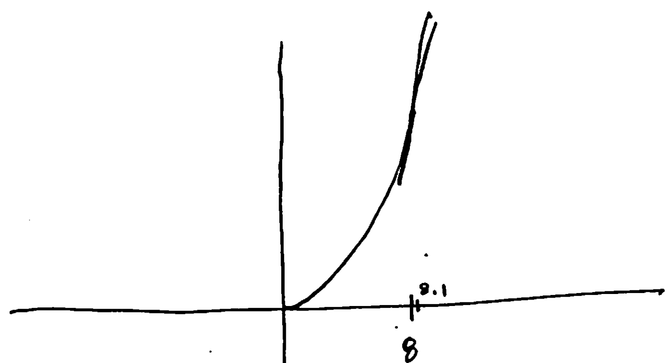
$df = (3 \cdot 8^2)(.1)$

$df = (192)(.1) = 19.2 \approx \Delta f$

$$\begin{array}{r} 8.1 \\ 8.1 \\ \hline \\ \hline 8.1 \end{array}$$

check:

$(8.1)^3 - (8)^3 = 19.441$



relative change: $\dfrac{df}{f} = \dfrac{19.2}{8^3}$

$\approx \dfrac{19.2}{512} \approx .0375$

3.75%

Sphere:

$$V = \frac{4}{3}\pi r^3$$

$$V' = \frac{4}{\cancel{3}}\pi \cdot (\cancel{3}r^2) = \underline{4\pi r^2}$$

$r = 5\,m$

$\Delta r = .003\,m$

$$dV = [V'(r)] \cdot \Delta r$$

$$dV = [V'(5)] \cdot (.003)$$

$$\rightarrow dV = [4\pi \cdot 5^2] \cdot (.003)$$

$$* \quad dV = (100\pi)(.003)\,m^3 = \underline{.9425\,m^3}$$

$$(264 \text{ gallons} = 1\,m^3)$$

$$(.9425)(264) \approx 249 \text{ gallons}$$

exact: $\frac{4}{3}\pi(5.003)^3 - \frac{4}{3}\pi(5)^3$

$$dV = [V'(r)] \cdot \Delta r$$

$$\int \underline{dV} = \int [V'(r)] \cdot dr$$

$$V = \int \underline{V'(r)}\, dr$$

$$\int 3x^2 \cdot dx = x^3 + C$$ indefinite integral

integral sign (ANTIDERIVATIVE)

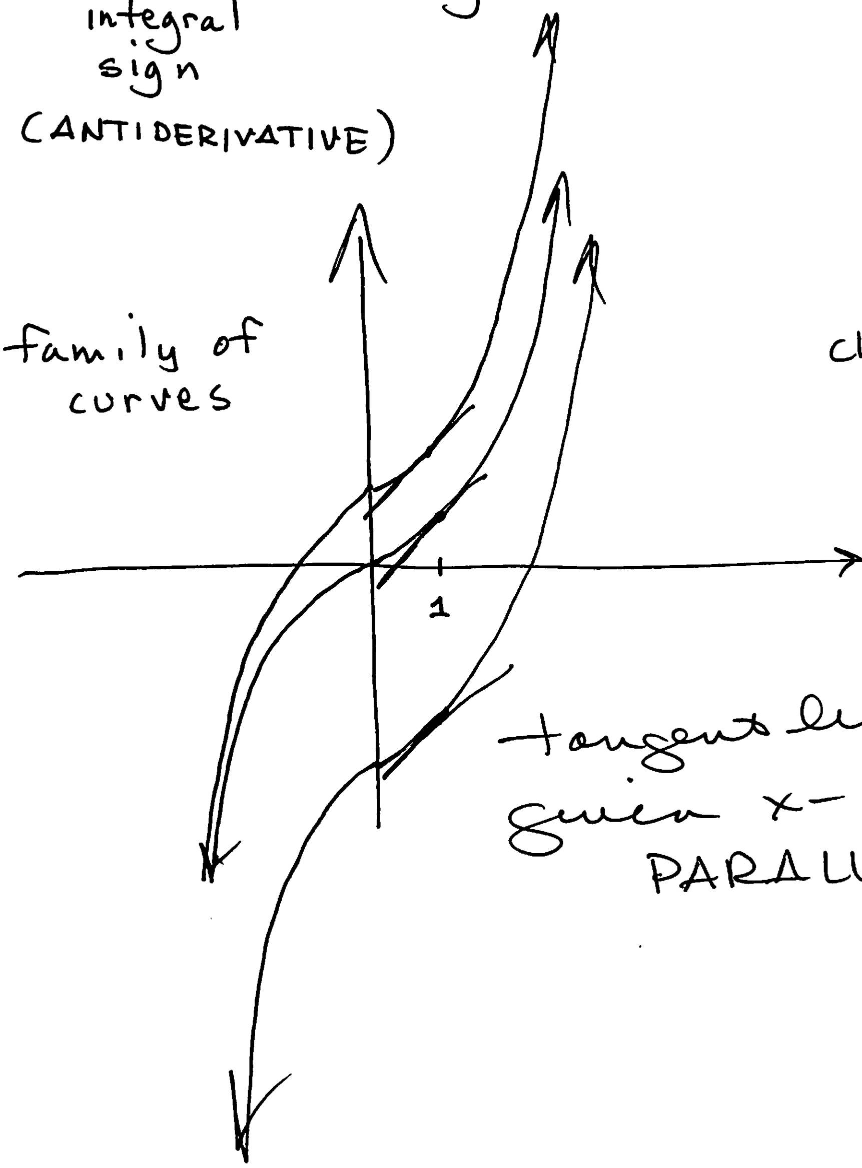
integrand

$$\int 3x^2 \cdot dx = x^3 + 1$$

$$\int 3x^2\, dx = x^3 - 17$$

$$\int 3x^2\, dx = x^3 + \frac{\pi}{e}$$

family of curves



check:

$$d(x^3+1) \stackrel{??}{=} 3x^2$$

$$d(x^3-17) \stackrel{??}{=} 3x^2$$

$$d(x^3+C) = 3x^2$$

tangent lines (for any given x-value) are PARALLEL

$$\int 4x^7\,dx = 4\cdot\frac{x^8}{8} + C$$

check: $d\left(4\cdot x^8 + C\right) \overset{??}{=} 4x^7$

$$= 4\cdot 8x^7 + 0$$

$$\int 4x^7\,dx = \frac{1}{2}x^8 + C$$

check: $d\left(\frac{1}{2}x^8 + C\right) \overset{??}{=} \frac{1}{2}\cdot 8\cdot x^7 + 0$

$$= 4x^7$$

GEN.

$$\boxed{\int a\cdot x^n\,dx = a\cdot\frac{x^{n+1}}{n+1} + C}$$ for $n \neq -1$

check: $d\left[\frac{a}{n+1}\boxed{x^{n+1}} + C\right] = \frac{a}{\cancel{n+1}}\cdot(\cancel{n+1})\,x^n + 0$

$$= a\cdot x^n$$

ex: $\int 3\cdot x^5\,dx = 3\cdot\frac{x^6}{6} + C$

$$= \frac{1}{2}x^6 + C$$

$$\int 7 \cdot x^{-1/2}\,dx$$

$$= 7 \cdot \frac{x^{1/2}}{1/2} + C$$

$$= \underline{14 \cdot x^{1/2} + C}$$

check:

$$d(14x^{1/2}+c) \overset{??}{=} 7x^{-1/2}$$

$$\int 11 \cdot x^{-4}\,dx = 11 \cdot \frac{x^{-3}}{-3} + C$$

$$= -\frac{11}{3}x^{-3} + C$$

$$\int 3\,dx$$

$$= 3x + C$$

$$\int 3\,dt$$

$$= 3t + C$$

$$\int 3v^3 \cdot dv$$

when $n = -1$

$$\int a \cdot x^n \cdot dx = \int a \cdot x^{-1} \cdot dx$$

$$= a \int x^{-1} dx = a \int \frac{1}{x} \cdot dx$$

$$= a \cdot \ln|x| + C$$

$$\int e^x \cdot dx = e^x + C$$

$$\int e^{8x} dx = \frac{1}{8} e^{8x} + C$$

check: $d\left(\frac{1}{8} \cdot e^{8x} + c\right) \overset{??}{=} e^{8x}$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$(\arctan x + c)$

8

$$\int f(x)\cdot dx = F(x) + C$$

(where $F'(x) = f(x)$)

$$\int \left(3x^3 + 8x^2 - 14x + \frac{9}{x} - 11\right)dx$$

$$= \int 3x^3\,dx + \int 8x^2\,dx - \int 14x\,dx$$

$$+ \int \frac{9}{x}\,dx - \int 11\,dx$$

$$= \underbrace{3\int x^3\,dx} + \underbrace{8\int x^2\,dx} \underbrace{- 14\int x\,dx}$$

$$+ 9\int \frac{1}{x}\,dx - 11\int 1\,dx$$

$$\int dx$$

$$= 3\cdot\frac{x^4}{4} + 8\cdot\frac{x^3}{3} - 14\frac{x^2}{2} + 9\ln|x| - 11x + C$$