MA141-004

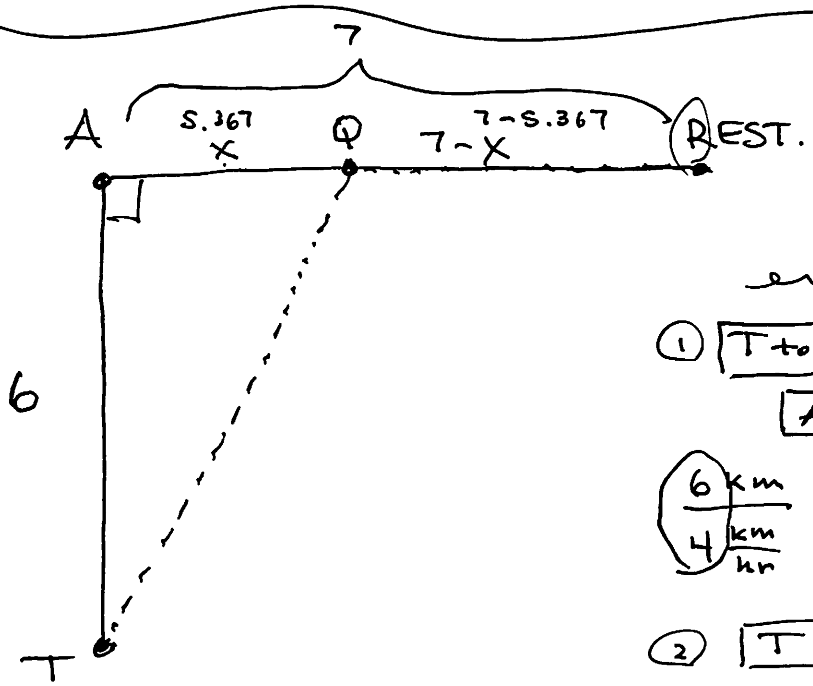
Thursday, March 7

today:

- finish optimization problems (3.4)
- begin L'Hopital's Rule (3.5)

single var.

15.)



$D = r \cdot t$

$\frac{D}{r} = t$

endpts

① $\boxed{T \text{ to } A}$ then $\boxed{A \text{ to } R}$

$\frac{6 \text{ km}}{4 \frac{\text{km}}{\text{hr}}} + \frac{7 \text{ km}}{6 \frac{\text{km}}{\text{hr}}} = \underline{2.67 \text{ hr}}$

② $\boxed{T \text{ to } R}$

$\frac{\sqrt{6^2+7^2} \text{ km}}{4 \frac{\text{km}}{\text{hr}}} = \underline{2.305 \text{ hr}}$

CALC: (MIN TIME)

$\boxed{TQ}$

$$\frac{\sqrt{x^2+6^2} \text{ km}}{4 \frac{\text{km}}{\text{hr}}} + \frac{7-x \text{ km}}{6 \frac{\text{km}}{\text{hr}}} = T$$

$$\frac{1}{4}\boxed{(x^2+36)^{1/2}} + \frac{1}{6}\boxed{(7-x)} = T$$

$$\frac{1}{4} \cdot \frac{1}{\cancel{2}}(x^2+36)^{-1/2} \cdot (\cancel{2}x) + \frac{1}{6} \cdot (-1) = T' = 0$$

$$\frac{x}{4 \cdot \sqrt{x^2+36}} + \frac{-1}{6} = 0$$

$$\frac{x}{4\sqrt{x^2+36}} = \frac{1}{6}$$

$$6x = 4\sqrt{x^2+36}$$

$$36x^2 = 16\cdot(x^2+36)$$
$$36x^2 = 16x^2 + (16)(36)$$
$$-16x^2 \quad -16x^2$$

$$\frac{\cancel{20}x^2}{\cancel{20}} = \frac{(16)(36)}{20}$$

$$x^2 = \frac{(16)(36)}{20}$$

$x \approx 5.367$ ~~km~~ km

max or min ??

$$T'' = \text{deriv}\left(\frac{1}{4}\cdot x\cdot(x^2+36)^{-1/2}\right)$$

$$T'' = \frac{1}{4}\cdot x\cdot \frac{-1}{2}(x^2+36)^{-3/2}(2x) + (x^2+36)^{-1/2}\cdot\frac{1}{4}$$

$$T''(5.367) = \frac{1}{4}(5.367)\cdot\frac{-1}{2}((5.367)^2+36)^{-3/2}(2(5.367)) + \frac{1}{4}(5.367^2+36)^{-1/2} = +$$

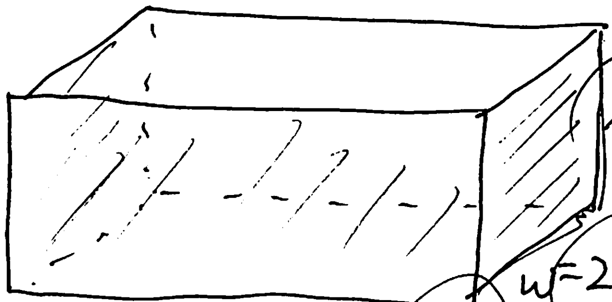
$\therefore$ C.UP

$\therefore$ MIN

$$T = \frac{\sqrt{5.367^2+36}}{4} + \frac{7-5.367}{6} \approx \underline{2.28 \text{ hr}}$$

9.)

$V = 18\ yd^3$



$\cancel{H}\ \frac{6}{w^2} = \frac{6}{4}\ yds$

$w = 2\ yd$

single var

$\cancel{L}\ 3w = 6\ yd$

$(3w)(w)(H) = 18$

$3w^2 H = 18$

$H = \frac{18}{3w^2} = \frac{6}{w^2}$

check:

$\left(\frac{6}{\cancel{4}_2}\right)(\cancel{2}^1)(6) = \underline{\underline{18}}$

$S = 2(3w)\cdot\left(\frac{6}{w^2}\right) + 2(w)\left(\frac{6}{w^2}\right)$

$+ (3w)(w)$

$S = \frac{36}{w} + \frac{12}{w} + 3w^2$

$S = \frac{48}{w} + 3w^2$

$48w^{-1}$

$S' = \frac{-48}{w^2} + 6w = 0$

$\frac{6w}{1} = \frac{+48}{w^2}$

$48 = 6w^3$

$8 = w^3$

$2 = w$

$\left\{ \begin{array}{l} S'' = \frac{96}{w^3} + 6 = + \\ \therefore \text{c. up} \\ \therefore \underline{\underline{\text{MIN}}} \end{array} \right.$

3.5: $\lim_{x \to a} \frac{f(x)}{g(x)} = ??$ ← indeterminate forms

$\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$

APPLICATION: L'HOPITAL'S RULE

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \sim \xrightarrow{\text{L'HOP}} \lim_{x \to a} \frac{f'(x)}{g'(x)} = \sim$$

$$\xrightarrow{\text{L'HOP}} \lim_{x \to a} \frac{f''(x)}{g''(x)} \quad \ldots$$

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ex:

$$\lim_{x \to 3} \frac{x-3}{x^2-9} \xrightarrow{\text{L'HOP}} \lim_{x \to 3} \frac{1}{2x} = \frac{1}{6}$$

$$\lim_{x \to 3} \frac{x-3}{x^2-9} = \lim_{x \to 3} \frac{\cancel{x-3}^{\,1}}{\cancel{(x-3)}_{1}(x+3)} = \lim_{x \to 3} \frac{1}{(x+3} = \frac{1}{6}$$

ex: $\lim\limits_{x\to 0} \dfrac{16x^2}{5\cos x - 5} \xrightarrow{\text{L'HOP}} \lim\limits_{x\to 0} \dfrac{32x}{-5\sin x}$

$\xrightarrow{\text{L'HOP}} \lim\limits_{x\to 0} \dfrac{32}{-5\cos x} = \dfrac{-32}{5}$

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ex: $\lim\limits_{x\to\infty} \dfrac{\ln(2x)}{5x^2} \xrightarrow{\text{L'HOP}} \lim\limits_{x\to\infty} \dfrac{\frac{1}{\cancel{2}x}(\cancel{2})}{10x}$

$\lim\limits_{x\to\infty} \left(\dfrac{1}{x}\cdot\dfrac{1}{10x}\right) = \lim\limits_{x\to\infty} \dfrac{1}{10x^2} = 0$

"other" indeterminate forms

$(0\cdot\infty$ and $\infty-\infty)$ $\qquad 0\cdot\infty \stackrel{?}{=} 5$

$\lim\limits_{x\to 0} \left(x\cdot\left(\dfrac{5}{x}\right)\right) = 5$

$0\cdot\infty$

* rearrange so as to have a "standard" indet. form $\frac{0}{0}$ $\frac{\infty}{\infty}$

ex: $\lim\limits_{x\to 0^+} \boxed{x\cdot \ln x} = \lim\limits_{x\to 0^+} \boxed{\dfrac{\ln x}{\frac{1}{x}}}$ x^{-1}

$\xrightarrow{\text{L'HOP}} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim\limits_{x\to 0^+} \dfrac{1}{x}\cdot\dfrac{x^2}{-1}$

$= \lim\limits_{x\to 0^+} (-1\cdot x) = 0$

$\underline{\infty - \infty}$

ex: $\lim_{x\to\infty} \left(x e^{1/x} - x \right)$

$= \lim_{x\to\infty} x\left(e^{1/x} - 1\right) = \lim_{x\to\infty} \dfrac{e^{1/x} - 1}{\frac{1}{x}}$

$\xrightarrow{\text{L'HOP}} \lim_{x\to\infty} \dfrac{\boxed{e^{1/x}} \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = 1$

$\left(\underline{1^\infty}; \ \infty^0; \ 0^0 \right)$

$x = 10{,}000$

$\underbrace{\left(1 + \frac{1}{10{,}000}\right)}^{10{,}000} \approx \underline{2.718}$

ex: $\lim_{x\to\infty} \boxed{\left(1 + \frac{1}{x}\right)^x} =$ rewrite using $e^{\ln \alpha} = \alpha$

$e^{\ln u} = u$

$e^{\ln\left(1+\frac{1}{x}\right)^x} = \left(1 + \frac{1}{x}\right)^x$

$\ln 2^3 = 3 \cdot \ln 2$

$\lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x}$

$\lim_{x\to\infty} e^{\boxed{x \cdot \ln\left(1+\frac{1}{x}\right)}} \leftarrow 1 \qquad \therefore e^1 = e$

$\lim_{x\to\infty} x \cdot \ln\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty} \dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$\xrightarrow{\text{L'HOP}} \lim_{x\to\infty} \dfrac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = 1$

$$\lim_{x\to 0^+} x^x$$

$$\lim_{x\to 0^+} e^{\ln x^x}$$

$$\lim_{x\to 0^+} e^{\boxed{x\cdot \ln x}}$$

$$\lim_{x\to 0^+}[x\cdot \ln x] = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$\xrightarrow{\text{L'HOP}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to 0^+}\left(\frac{1}{x}\cdot\frac{x^2}{-1}\right) = \lim_{x\to 0^+}(-1\cdot x)$$

$$= 0$$

$$\therefore e^0 = 1$$

MA 141-004 S19 Recitation 9, Mar 6

1. Find the absolute minimum and maximum values of $f(x) = \frac{1}{4}x^4 - 2x^2$ on the closed interval $[-1, 3]$. endpoints $(-1, ?)$ & $(3, ?)$

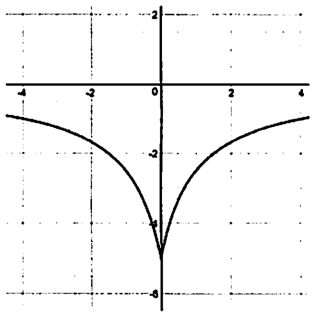
2. The function $f(x) = \frac{10x}{x^2+1}$ is continuous and differentiable on $\mathbb{R}$. $f'(x) = \frac{\quad}{(x^2+1)^2}$

   (a) Use critical numbers to determine where f is increasing and where it is decreasing.

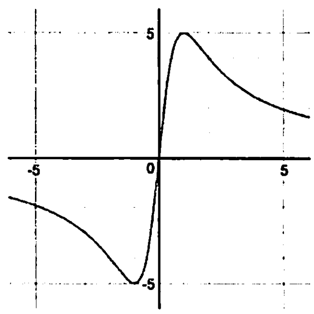
   (b) Does $f(x)$ have any local minimum or local maximum points?

   (c) Which of the following is the graph of $f(x)$?

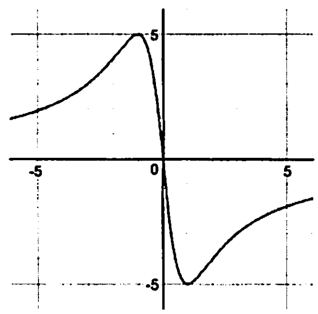
(i)



(ii)



(iii)



3. Show that the function $g(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + 3x + 4$ has one inflection point at $(1, \frac{20}{3})$.

4. The position of a particle moving along the x-axis is given by $s(t) = t(5-t)$ where $t \in [0, 5]$ (in seconds).

   (a) True or False: By Rolle's Theorem there is some time $t_0 \in (0, 5)$ when the velocity of the particle is zero.

   (b) True or False: By Mean Value Theorem there is some time $t_0 \in (0, 1)$ when the velocity of the particle is 4.

Procedure to Find Global Maximum and Minimum of f on $[a, b]$:

Assume that f is continuous on the closed and bounded interval $[a, b]$.

1. Find the x- and y-coordinates $(x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)), \cdots$ of the points with x-coordinates in $[a, b]$ where f attains local extreme values. Create the set of y-values $S = \{f(x_1), f(x_2), \cdots\}$.
2. Add the values $f(a)$ and $f(b)$ to the set S:

$$S = \{f(a), f(b), f(x_1), f(x_2), \cdots\}$$

3. The largest number in the set S is the absolute maximum of f on the interval $[a, b]$, and the smallest number in the set S is the absolute minimum value of f on $[a, b]$.

Theorem 3. Rolle's Theorem
Let f be a function that is continuous on the closed, bounded interval $[a, b]$, differentiable on the open interval (a, b), and suppose that $f(a) = f(b)$. Then there is at least one point $c \in (a, b)$ where $f'(c) = 0$.

Theorem 4. The Mean Value Theorem
Let the function f be continuous on the closed and bounded interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$