MA 141-004 

Tuesday, March 5

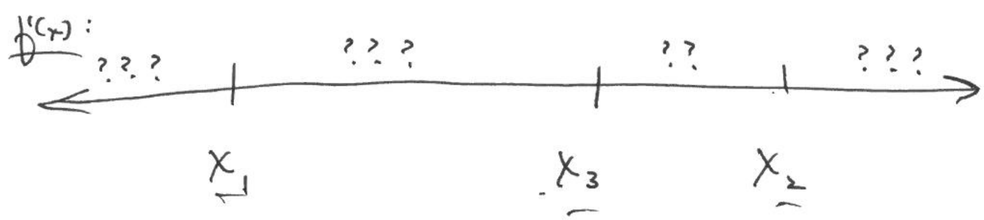
## 3.3: using $y'$ & $y''$ to graph

### $y'$ INFO:

$$\begin{cases} \text{if } f'(x) = +, \text{ then } f(x) \text{ is INCR.} \\ \text{if } f'(x) = -, \text{ then } f(x) \text{ is DECR.} \end{cases}$$

CRITICAL VALUES:

1. $f'(x) = 0$ "FLAT" ✓
2. $f'(x)$ undef "STEEP" ✓




ex: $f(x) = x^3 + 3x^2 + 2$

$$\boxed{f'(x) = 3x^2 + 6x} + 0$$

~~$3x^2+6x$ undef?~~

$$3x^2 + 6x = 0$$

$$3x(x+2) = 0$$

$(0,2)$ ← $x = 0$ or $x + 2 = 0$

$\boxed{x = 0}$ $\boxed{x = -2}$ → $(-2, 6)$

CRITICALS: $f(0) = 2$

$f(-2) = (-2)^3 + 3(-2)^2 + 2 = 6$

**Definition 6. Increasing and Decreasing**

*Let the function $f$ be defined on an interval $I$.*

*$f$ is* **increasing on** *$I$ (also called* **strictly increasing***) if, for $x_1, x_2 \in I$,*

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2$$

*Similarly, $f$ is* **decreasing on** *$I$ (also called* **strictly decreasing***) if, for $x_1, x_2 \in I$,*

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2$$

**Theorem 1. Increasing and Decreasing on a Closed Interval**

*Let the function $f$ be continuous on the closed, bounded interval $[a, b]$ and differentiable on the open interval $(a, b)$. Then*

1. *If $f'(x) > 0$ on $(a, b)$, then $f$ is increasing on $[a, b]$.*
2. *If $f'(x) < 0$ on $(a, b)$, then $f$ is decreasing on $[a, b]$.*

**Theorem 2. Increasing and Decreasing on an Open Interval**

*Let the function $f$ be differentiable on the open interval $(a, b)$. Then*

1. *If $f'(x) > 0$ on $(a, b)$, then $f$ is increasing on $(a, b)$.*
2. *If $f'(x) < 0$ on $(a, b)$, then $f$ is decreasing on $(a, b)$.*

**Definition 8. Concavity**

*Let the function $f$ be differentiable on the interval $(a, b)$.*

($f''(x) = +$)

1. *If $f'$ is increasing on $(a, b)$, then the graph of $f$ is* **concave up** *on $(a, b)$.*
2. *If $f'$ is decreasing on $(a, b)$, then the graph of $f$ is* **concave down** *on $(a, b)$.*

($f''(x) = -$)

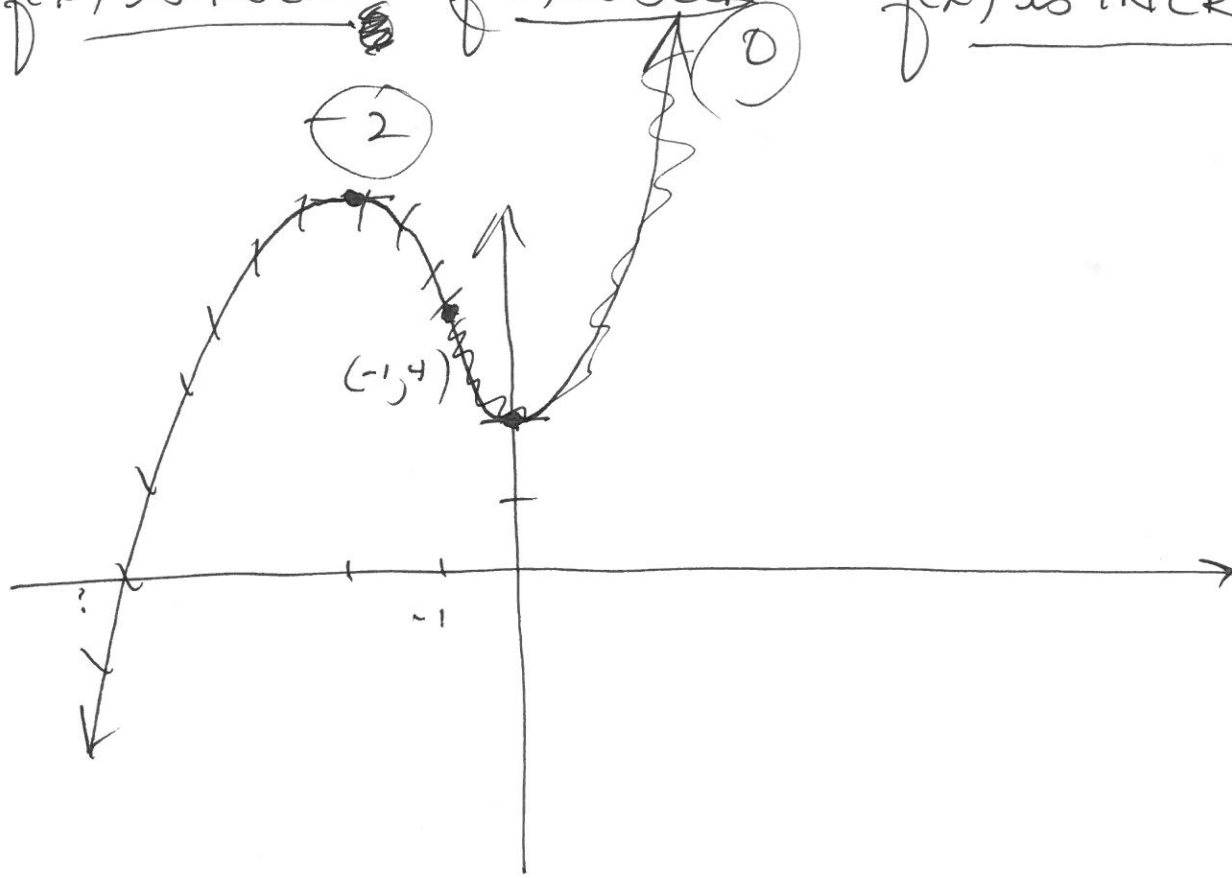
**Example 25.** *When the ticket price is \$40, the average attendance at the football game is 40,000 people. It has been determined that for every \$1 decrease in the ticket price, an additional 2000 people will purchase tickets and attend the game. Under this arrangement, what price should be charged per ticket to maximize the revenue for the university? How many fans will attend the game at this price? What is the maximum revenue?*

$f'(x) = \underbrace{3x}\underbrace{(x+2)}$

$\underline{f'(x):}$

$f'(-3) = - \cdot - = +$ | $f'(-1) = - \cdot + = -$ | $f'(1) = + \cdot + = +$

$f(x)$ is INCR. | $f(x)$ is DECR | $f(x)$ is INCR.



ex: not a polynom.

$$y = \underbrace{1}_{} + \underbrace{(x-3)^{2/3}}_{}$$

$$y' = 0 + \frac{2}{3}(x-3)^{-1/3}(1)$$

$$y' = \frac{2}{3(x-3)^{1/3}} = \frac{2}{3 \cdot \sqrt[3]{x-3}}$$

① $y' = 0$

$$\frac{2}{3\sqrt[3]{x-3}} \neq 0$$

no "flat" places

② $y'$ undef.

$$\frac{2}{3 \cdot \sqrt[3]{x-3}} \text{ undef.}$$

$$3 \cdot \sqrt[3]{x-3} = 0$$

when $x = 3$

(vertical tangent line there)

$$\underline{(3, 1)} = (3, f(3))$$

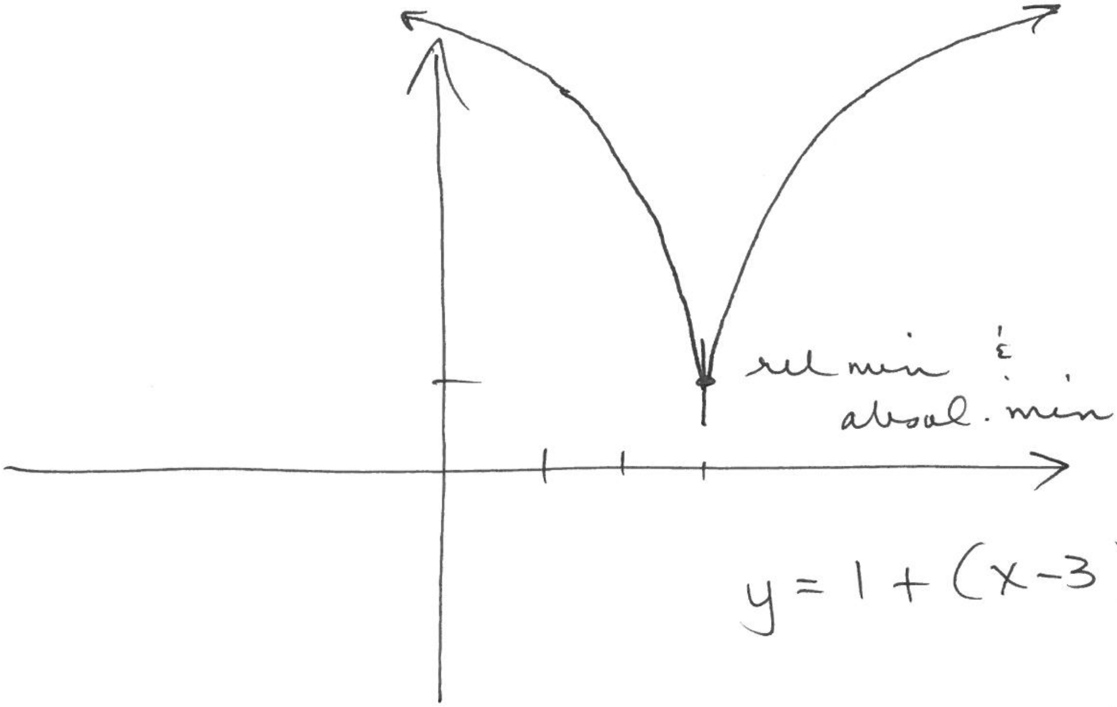
$$f(3) = 1 + (3-3)^{2/3} = 1$$

$$f'(x) = \frac{2}{3 \cdot \sqrt[3]{x-3}}$$

$f'(x)$:

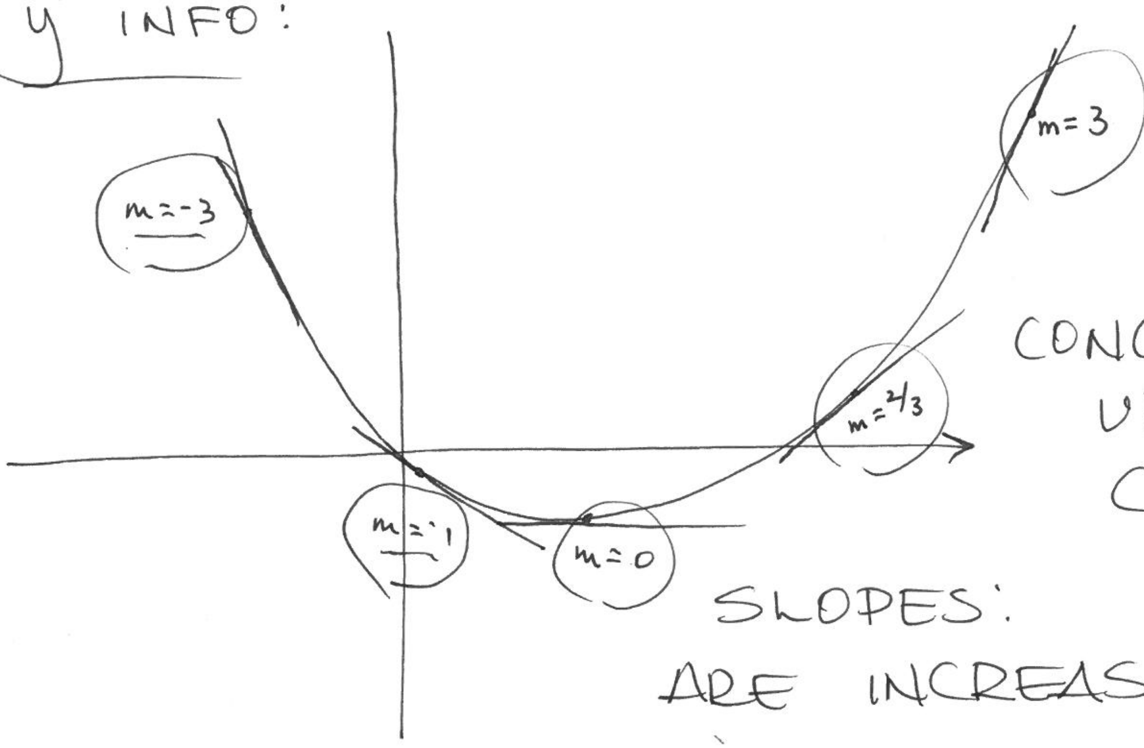
$f'(0) = \frac{+}{+ \cdot -} = -$ | $f'(4) = \frac{+}{+ \cdot +} = +$

$f(x)$ DECR. | 3 | $f(x)$ INCR




$y = 1 + (x-3)^{2/3}$

y'' INFO:



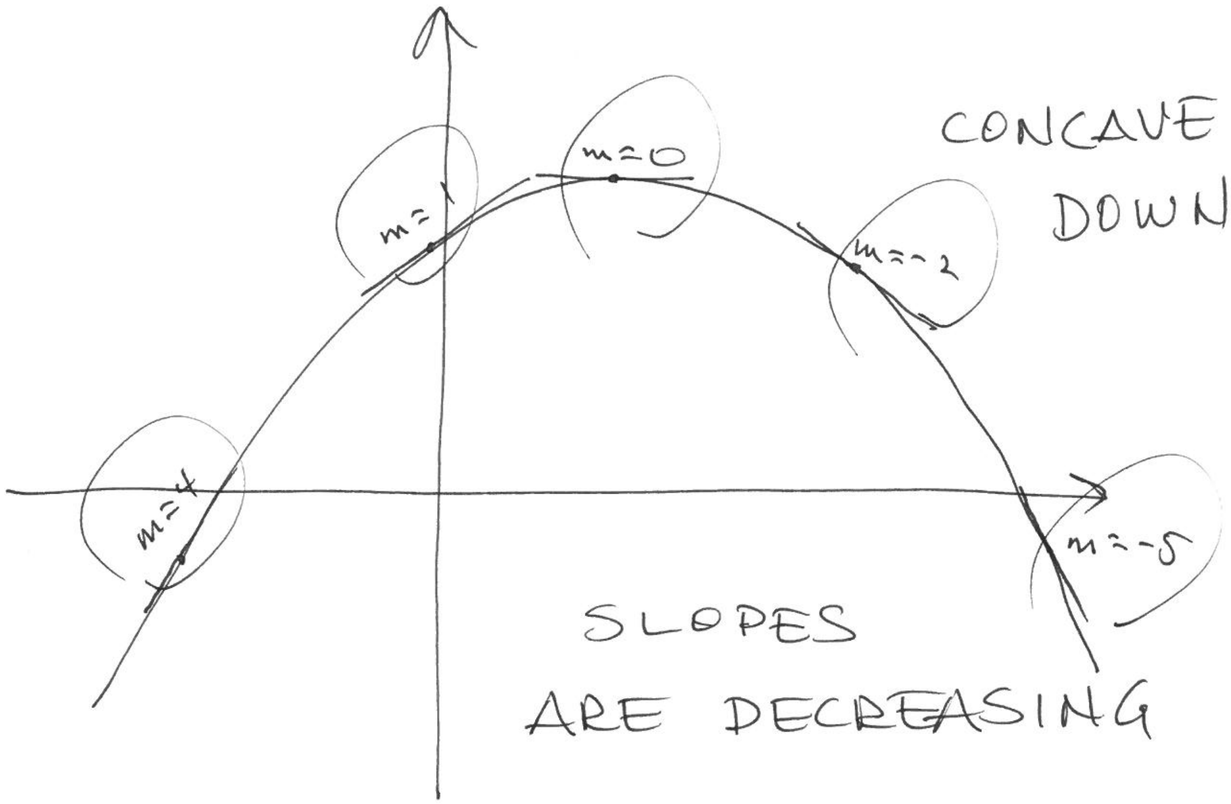

CONCAVE UP (C.UP)

SLOPES: ARE INCREASING

if slopes ($\to f'(x)$) are INCR, then $f(x)$ is CONCAVE UP

if $f''(x) > 0$, then $f'(x)$ INCR, then $f(x)$ is C.UP

~~A~~ if $f''(x) > 0$, then $f(x)$ is CONCAVE UP.




if $f''(x) < 0$, then $f'(x)$ is DECR.

if $f'(x)$ DECR, then $f(x)$ is C. DOWN

~~A~~ if $f''(x) < 0$, then $f(x)$ is CONCAVE DOWN

$f(x) = x^3 + 3x^2 + 2$

$f'(x) = 3x^2 + 6x$

$f''(x) = 6x + 6$

(1) $f''(x) = 0$

(2) $f''(x)$ undef.

$\boxed{6x+6} = 0$

$6x = -6$

$x = -1$

$(-1, f(-1))$

$f(-1) = (-1)^3 + 3(-1)^2 + 2$
$= -1 + 3 + 2$
$= 4$

$(-1, 4)$

$f''(x) = 6x + 6$

$f''(x)$:

$f''(-2) = -$ | $f''(0) = +$

$f(x)$ CONCAVE DOWN | $-1$ | $f(x)$ CONCAVE UP

$(-1, 4)$ → changed CONCAVITY there
point of inflection

$$y = 1 + (x-3)^{2/3}$$

$$y' = \frac{2}{3}(x-3)^{-1/3}$$

$$y'' = \frac{2}{3}\left[-\frac{1}{3}\cdot(x-3)^{-4/3}(1)\right]$$

$$y'' = \boxed{\frac{-2}{9\cdot(x-3)^{4/3}}} = \frac{-2}{9\cdot\left[\sqrt[3]{x-3}\right]^4} = \frac{-}{+\cdot+} = -$$

① $\cancel{y''=0}$ ② $y''$ undef at $x=3$

$\therefore$ always C. DOWN

$f''(x)$:

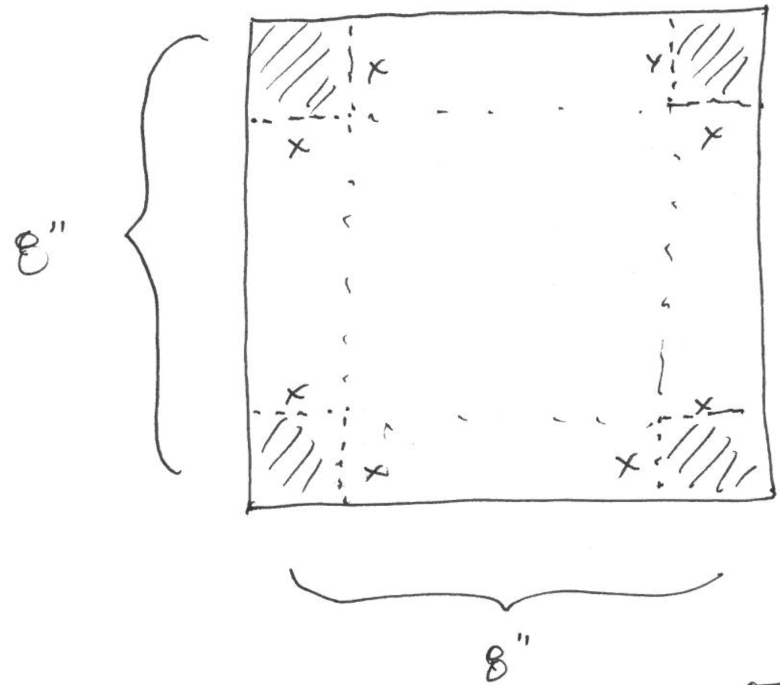
$f''(0) = -$ | $3$ | $f''(4) = -$

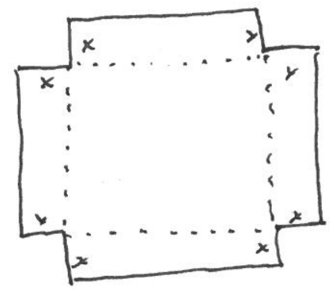
$f(x)$ C. DOWN | $3$ | $f(x)$ C. DOWN

3.4:

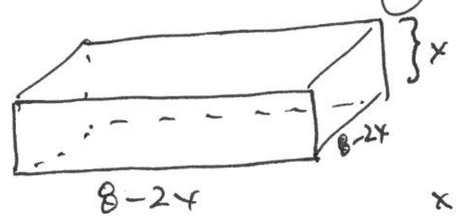
# OPTIMIZATION

## (MAX / MIN WORD PROBLEMS)




form a box of MAX VOL




$x=0$
$x=1$
$x=2$

$$V = \underbrace{(8-2x)(8-2x)}\,x$$

$$V(x) = (64 - 32x + 4x^2)\,x$$

$$V(x) = \underline{64x}\ \underline{-32x^2}\ \underline{+4x^3} \qquad (V'(x)=0)$$

$$V'(x) = \boxed{64 - 64x + 12x^2 = 0}$$

$$4(3x^2 - 16x + 16) = 0$$

$$4\,(3x-4)(x-4) = 0$$

$$0 < x < 4$$

$3x-4=0$ or $x-4=0$

$\underbrace{x = 4/3}$ or $\cancel{x=4}$

max or min

$\therefore$ C. DOWN
$\therefore$ MAX

VALIDATION:

$$V''(x) = 0 - 64 + 24x \qquad V''\left(\tfrac{4}{3}\right) = -64 + 24\left(\tfrac{4}{3}\right)$$

$$V''\left(\tfrac{4}{3}\right) = -$$

Since $r > 0$ we consider only the positive root.

$$r_1 = \sqrt{\frac{A_0}{6\pi}}$$

The second derivative $V''(r) = -6\pi r$ is negative on the interval $(0, \infty)$. Since this interval is our domain of interest, we can apply Theorem 1 to conclude that $r_1$ is the only critical number of interest and that it is the location of the global maximum. Hence the global maximum volume is

$$V_{max} = V(r_1) = -\pi(r_1)^3 + \frac{1}{2}A_0 r_1 = -\pi\left(\sqrt{\frac{A_0}{6\pi}}\right)^3 + \frac{1}{2}A_0\sqrt{\frac{A_0}{6\pi}}$$

Using this value we find the corresponding $h_1$ to be

$$h_1 = \frac{A_0 - 2\pi r_1^2}{2\pi r_1} = \frac{A_0 - 2\pi\left(\sqrt{\frac{A_0}{6\pi}}\right)^2}{2\pi\sqrt{\frac{A_0}{6\pi}}} = \sqrt{\frac{2A_0}{3\pi}} = \sqrt{\frac{2}{2}\frac{2A_0}{3\pi}} = \sqrt{\frac{4A_0}{6\pi}} = 2\sqrt{\frac{A_0}{6\pi}} = 2r_1$$

Thus the volume of a cylinder is maximized for a fixed surface area when the height is twice the radius.

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**Example 25.** *When the ticket price is \$40, the average attendance at the football game is 40,000 people. It has been determined that for every \$1 decrease in the ticket price, an additional 2000 people will purchase tickets and attend the game. Under this arrangement, what price should be charged per ticket to maximize the revenue for the university? How many fans will attend the game at this price? What is the maximum revenue?*

**Solution:** We need to pick a variable. While the new ticket price and the number of attendees are important items in this problem, let's choose $x$ to be the amount we will reduce the ticket price. Then the new ticket price is $40 - x$ and the number of people attending is $40,000 + 2,000x$. The revenue $R$ is the product of the ticket price and and the number of attendees.

$$R(x) = (40 - x)(40,000 + 2000x)$$

To find the critical points we differentiate $R$, set $R'$ equal to zero, and solve for $x$.

$$R'(x) = -1(40,000 + 2000x) + 2000(40 - x) = 40,000 - 4,000x = 0$$

$$x = 10$$

Since $R'$ is continuous and has only one zero, we determine that it is positive on $(-\infty, 10)$ and negative on $(10, \infty)$ by evaluation at two points, say $R'(0) = 40,000$ and $R'(11) = -4,000$. Hence by the first derivative test we have the global maximum revenue when we reduce the price by \$10. The price of the ticket should be set at $\$40 - \$10 = \$30$. The total attendance will be $40,000 + 2000 \cdot 10 = 60,000$ and the maximum revenue will be

$$R(10) = 30 \cdot 60,000 = \$1,800,000.$$

The revenue when the ticket price is \$40 is \$1,600,000; so the increase in revenue would be \$200,000.

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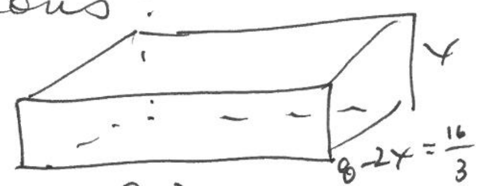
dimensions?




$$x = 4/3 \qquad \frac{8-2x}{8 - 8/3 = \frac{16}{3}}$$

DIM: $\left(4/3 \text{ in}\right)$ by $\left(16/3 \text{ in}\right)$ by $\left(16/3 \text{ in}\right)$

max volume?

$$V = \left(\frac{4}{3}\right)\left(\frac{16}{3}\right)\left(\frac{16}{3}\right) = \frac{1024}{27}\ \text{in}^3$$

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let x = # of times to reduce the price of the ticket

$$REV = \left(\underline{40^{\underline{00}} - 1 \cdot x}\right)\left(40{,}000 + 2{,}000 \cdot x\right)$$

↑ ticket price ↑ attendance

$$R(x) = (40 - x)(40{,}000 + 2{,}000x)$$

$$R(x) = 1{,}600{,}000 + 80{,}000x - 40{,}000x - 2{,}000x^2$$

$$\boxed{R(x) = 1{,}600{,}000 + 40{,}000x - 2{,}000x^2}$$

$$R'(x) = 0 + 40{,}000 - 4000x = 0$$

$R''(x) = -4000$ ∴ NEG ∴ CONC. D ∴ MAX $\quad \dfrac{40{,}000}{4{,}000} = \dfrac{4000x}{4000}$ $x = 10$

price: $(40 - x) \rightarrow (40 - 10) = \$30^{\underline{00}}$

attendance: $(40{,}000 + 2{,}000x)$

$\longrightarrow (40{,}000 + 20{,}000)$

$= 60{,}000$ people

REV: $(30^{\underline{00}})(60{,}000) = \underline{\$1{,}800{,}000}$

$(29^{\underline{00}})(62{,}000) = 1{,}798{,}000$

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TEST #2 RESULTS

| | | |
|---|---|---|
| A's | 4 (13.3%) | 46.7% |
| B's | 10 (33.3%) | |
| C's | 7 (23.3%) | |
| D's | 3 (10%) | 30% |
| F's | 6 (20%) | |

AVE: 72.7