MA 141-005 ①

monday, march 18

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

$$0 = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} - \frac{f'(c) \cdot \Delta x}{\Delta x}$$

$$0 = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c) - f'(c) \cdot \Delta x}{\Delta x}$$

$$\Delta y = f'(c) \cdot \Delta x + \varepsilon(\Delta x) \cdot \Delta x$$

$$\Delta y = \left[f'(c) + \varepsilon(\Delta x)\right] \cdot \Delta x \quad * \leftarrow *$$

(x, x^3) $(x+\Delta x, (x+\Delta x)^3)$

ex: $f(x) = x^3$

$$\Delta y = (x+\Delta x)^3 - (x)^3$$

$$\Delta y = \cancel{x^3} + 3x^2 \cdot \Delta x + 3x \cdot (\Delta x)^2 + (\Delta x)^3 - \cancel{x^3}$$

$$\Delta y = (3x^2 + 3x \cdot \Delta x + (\Delta x)^2) \Delta x$$

$$\Delta y = \left[f'(x) + (3x \cdot \Delta x + (\Delta x)^2)\right] \cdot \Delta x$$

$$3x \cdot \Delta x + (\Delta x)^2 = \varepsilon(\Delta x)$$

as $\Delta x \to 0$ $\quad \varepsilon(\Delta x) \to 0$

$$\Delta y \approx dy = f'(\ ) \cdot \Delta x$$

(if Δx is small)

$f(x) = x^3$ from $x_1 = 8$ to $x_2 = 8.1$

$\Delta x = .1$

$$\Delta f \approx \boxed{\begin{aligned} df &= f'(x) \cdot \Delta x \\ &= 3x^2 \cdot \Delta x \\ &= 3(8)^2 \cdot (.1) \\ &= 19.2 \end{aligned}}$$

$$\Delta f = (8.1)^3 - (8)^3$$
$$= 531.441 - 512 = 19.441$$

$$\frac{df}{f} = \frac{19.2}{512} \approx \text{relative change}$$

$= .0375$

$= 3.75\%$ change

Example 38. *A spherical water tank of radius 5 meters is to be painted. The thickness of the paint on the sphere will be 3 mm. Use differentials to approximate the amount of paint in m^3 that it will take to paint the water tank. How many gallons of paint are required?*

Solution: The volume of the spherical tank is

$$V(r) = \frac{4}{3}\pi r^3$$

$$dV = V'(r)\,dr = (4\pi r^2)\,dr = (4\pi r^2)\,\Delta r$$

Using $r = 5$ m and $\Delta r = 3$ mm $= 0.003$ m yields

$$dV = (4\pi \cdot 5^2)\,0.003 \approx 0.9425\,\mathrm{m}^3$$

There are 264 gallons in 1 cubic meter. Hence it will take approximately 249 gallons of paint to cover the water tank.

$V = \frac{4}{3}\pi r^3$ $\quad V = \frac{4}{\cancel{3}} \cdot \pi \cdot \cancel{3} \cdot r^2$

$dy = f'(x) \cdot \Delta x$

$r = 5$ m

$\Delta r = .003$ m

$$\Delta V \approx \boxed{dV = [V'(r)] \cdot \Delta r}$$

$$= [4\pi \cdot r^2] \cdot \Delta r$$

$\frac{4}{3}\pi(5.003)^3 - \frac{4}{3}\pi(5)^3$

$$= [4\pi \cdot 5^2] \cdot (.003)$$

$$= (100\pi) \cdot (.003)$$

$$\approx \boxed{.9425}\ m^3$$

[264 gallons = 1 m^3]

$(.9425)(264) = 249$ gal.

$$dV = [V'(r)] \cdot \Delta r$$

$$\Delta V = \frac{dV}{dr} \cdot dr$$

$$\int dV = \int \frac{dV}{dr}\, dr$$

ANTIDERIVATIVE ....

$$V = \int V'(r)\, dr$$

$$\int \boxed{3}\, dr \quad \text{or} \quad \int \boxed{3}\, dt$$

$$= 3r + \cancel{8}\, C \qquad = 3t + C$$

check:

$$d(3r+8) \stackrel{?}{=} 3$$

$$d(3r-11) \stackrel{?}{=} 3$$

check:

$$d(3t+c) \stackrel{?}{=} 3$$

$$\int \underline{\qquad\qquad} \; dx$$

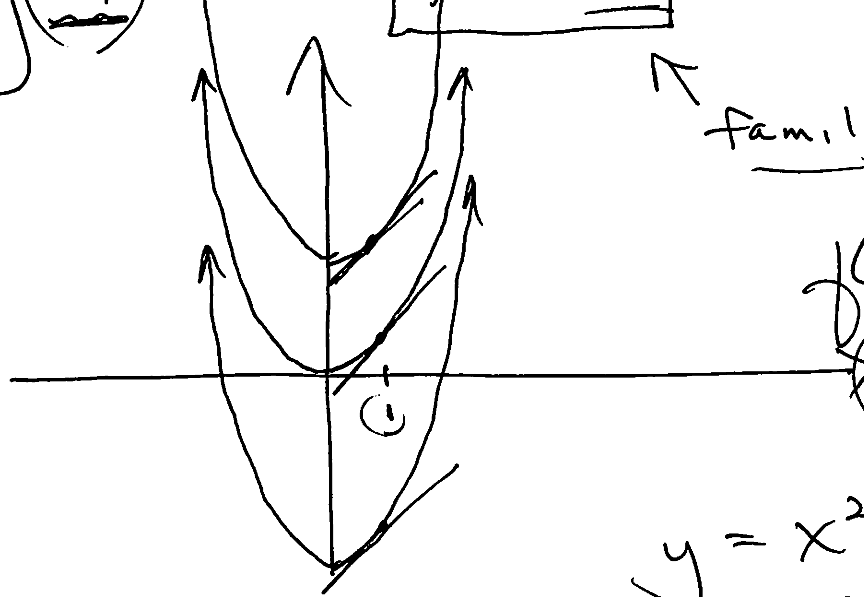
integrand

indefinite integrals

integral sign (antidifferentiate)

$$\int 2x \, dx = \boxed{x^2 + C}$$

family of curves



$f(1) = 6$

$x = 1$
$y = 6$

$$y = x^2 + C$$
$$6 = 1^2 + C$$
$$6 = 1 + C$$
$$C = 5$$
$$\therefore \; y = x^2 + 5$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int 2x \, dx = x^2 + C$$

$$\int 3x^2 \, dx = x^3 + C$$

$$\int x^4 \, dx = \frac{x^5}{5} + C$$

$$d\left(\tfrac{1}{5} \cdot x^5 + C\right) \overset{??}{=} x^4$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$
$$(\text{for } n \neq -1)$$

ex:

$$\int x^8\,dx = \frac{x^9}{9} + C$$

check: $d\left[\frac{1}{9}x^9 + C\right] \stackrel{?}{=} x^8$

$$\int x^{-3}\,dx = \frac{x^{-2}}{-2} + C$$

check: $d\left[-\frac{1}{2}(x^{-2}) + C\right] \stackrel{?}{=} -\cancel{\frac{1}{2}}\cdot(-\cancel{2}x^{-3})$

$= x^{-3}$

$$\int x^{1/2}\,dx = \frac{x^{3/2}}{3/2} + C$$

$d\left(\frac{2}{3}\cdot x^{3/2} + C\right) = \cancel{\frac{2}{3}}\cdot\cancel{\frac{3}{2}}\, x^{1/2} + 0$

$=$

---

when $n = -1$,

$$\int x^n\,dx = \int x^{-1}\,dx = \int \frac{1}{x}\,dx$$
$$= \ln|x| + C$$

$$\int 7x^{12}\,dx = 7\cdot\frac{x^{13}}{13} + C$$

$$\int \frac{8}{13}x^{-4/5}\,dx = \frac{8}{13}\cdot\frac{x^{1/5}}{1/5} + C$$

$$= \frac{8}{13}\cdot\frac{5}{1}x^{1/5} + C$$

$$= \frac{40}{13}x^{1/5} + C$$

$$\int e^{x}\cdot dx = e^{x} + C$$

$$\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

or

$$(\arctan x + C)$$

3.6 Differentials and Antiderivatives

Recall that Δy represents the total change of a function $y = f(x)$ as x varies from c to $c + \Delta x$ for some value c. We will use the definition of the derivative to split the **total change** Δy into two parts, the **principal part** which is related to $f'(x)$, and the **error part**. We begin with a theorem.

Theorem 1. *Let the function f be defined on the interval (a, b). If f is differentiable at $c \in (a, b)$ then*

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c) - f'(c)\Delta x}{\Delta x} = 0 \tag{11}$$

Proof. Although we can appeal to Lemma 1.3.1 to prove this formula let us show the limit exists by direct calculation. Thus suppose $f'(c)$ exists. Then

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c) - f'(c)\Delta x}{\Delta x} &= \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} - \lim_{\Delta x \to 0} \frac{f'(c)\Delta x}{\Delta x} \\ &= f'(c) - f'(c) \\ &= 0 \end{aligned}$$

□

We can conclude from this theorem that if a function f is differentiable at $x = c$, then the quantity that appears in Limit Formula (11) is a **function of** Δx, traditionally denoted $\varepsilon = \varepsilon(\Delta x)$:

$$\varepsilon = \frac{f(c + \Delta x) - f(c) - f'(c)\Delta x}{\Delta x} \tag{12}$$

By Theorem 1 the function ε has the property $\lim_{\Delta x \to 0} \varepsilon(\Delta x) = 0$. Solving Equation (12) for $\Delta y = f(c + \Delta x) - f(c)$ we have the following.

Theorem 2. *If f is differentiable at c then $\Delta y = f(c + \Delta x) - f(c)$ satisfies*

$$\underline{\Delta y = (f'(c) + \varepsilon)\Delta x} \tag{13}$$

if $\Delta x \to 0$

where the function ε has the property that

$\Delta y \doteq dy \approx f'(c) \cdot \underline{\Delta x}$

$$\lim_{\Delta x \to 0} \varepsilon(\Delta x) = 0$$

Last update: November 2, 2018

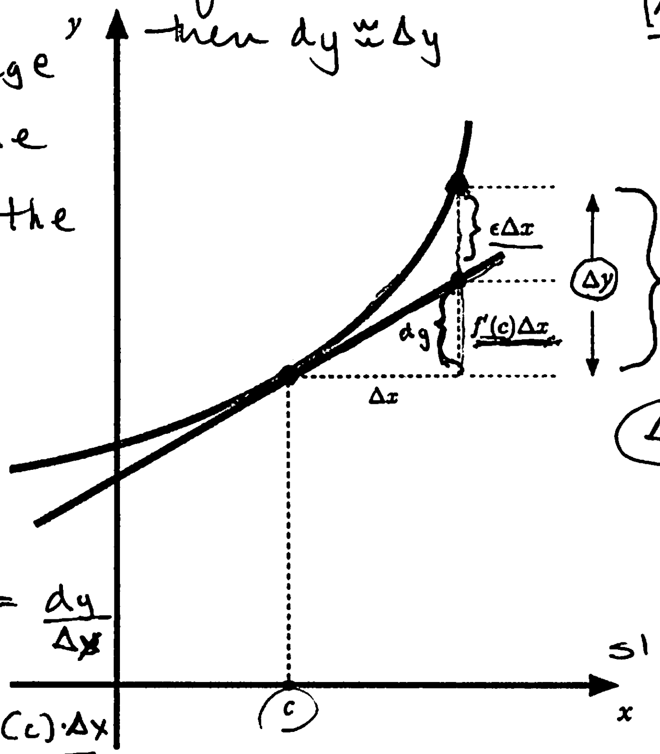
The term $f'(c)\Delta x$ in Equation (13) is called the **principal part** of Δy, since this is the part that is determined by the derivative $f'(c)$, and the term $\varepsilon\Delta x$ is called the **error part** of Δy. The two parts of Δy are shown in the following figure.

if Δx is "small" then $dy \approx \Delta y$

$\boxed{dy}$ = the change in y along the tangent to the curve

$dy = f'(c) \cdot \Delta x$

$\boxed{\Delta y}$ = exact change in y on the curve itself



$\Delta y = dy + \varepsilon \Delta x$

$\boxed{\Delta y = f'(c) \cdot \Delta x + \varepsilon \Delta x}$

$\Delta y = f'(c) \cdot \Delta x + \varepsilon(\Delta x) \cdot \Delta x$

$f'(c) = \frac{dy}{\Delta x}$

$dy = f'(c) \cdot \Delta x$

slope of tangent line: $f'(c)$

Figure 82: $\Delta y = f'(c)\Delta x + \varepsilon\Delta x$

REMARK: There are two notations involving ε and Δx which look very similar but have quite different meanings. This first comes from the definition of ε as a *function* of Δx. In this case we write $\varepsilon = \varepsilon(\Delta x)$. The second is the product of ε and Δx, $\varepsilon \cdot \Delta x = \varepsilon\Delta x$. This product is the *error* in using $dy = f'(c)\Delta x$ to approximate Δy.

Let's look at two examples to see what ε looks like in specific cases.

Example 34. *Let $f(x) = x^2$ with $f'(x) = 2x$. Find the function ε that appears in the Formula (13) above.*

